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THE RAMO-WOOLDRIDGE CORPORATION
Guided Missile Research Division

ON THE SLOSHING OF LIQUID IN A CYLINDRICAL TANK

by

John W. Miles

Report No. AM 6-5

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Aeromechanics Section

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M. V. Barton

SUMMARY

The free oscillations of a slightly viscous liquid in a cylindrical tank (of arbitrary cross section) with a free surface are analyzed and the decrement of motion caused by laminar boundary-layer friction and time-varying depth calculated. An increasing or decreasing (in time) depth is shown to have a destabilizing or stabilizing effect, respec .vely, but the magnitude of this effect is negligible for practical configurations. The results are applied to a circular tank, and it is shown that the depth may be considered infinite when it exceeds the diameter. The sloshing oscillations in a 10-foot (diameter) tank are found to have a (longest) period of 1.8 seconds and, assuming the kinematic viscosity to be that of water at 20°C, a time to damp to one-half amplitude of 7 minutes for mean amplitudes smaller than about 10 inches. The transverse, oscillatory force on the tank walls associated with a sloshing amplitude of one foot would have an amplitude of 2700 pounds and would act 2.7 feet below the mean surface.

A

SECTION I - INTRODUCTION

We shall consider the free oscillations of a slightly viscous fluid having a free surface but otherwise constrained by a vertical, cylindrical tank of varying (in time) depth. The preliminary analysis will be carried out for a cylinder of arbitrary cross section, after which the results will be applied to fuel sloshing in a circular tank.*

The effects of friction on the motion of the liquid are, by hypothesis, small and are confined to a thin boundary layer at the wall. It follows that a first approximation to the fluid motion, adequate except for the calculation of damping, may be calculated by neglecting friction entirely; this approximation then may be used as a basis for the calculation of the boundary layer flow. We find it expedient in this approach to aim at an energy formulation and toward this end consider the kinematics of this first approximation in Section II, establish the corresponding kinetic and potential energies in Section III, find the laminar boundary layer solution and corresponding dissipation function in Section IV, and calculate the frequency and decrement of the mean motion in Section V. The local equation of motion does not appear throughout this sequence but is introduced <u>a posteriori</u> in Section VI to calculate the dynamic forces on the tank walls. he results established in Sections II through VI are applied to rcular tank in Section VII and a numerical example considered.

*This problem also has been considered, from a somewhat different viewpoint, by R.R. Berlot in Ramo-Wooldridge Report GM-TN-2 (22 March 1956).

SECTION II - KINEMATIC FORMULATION

We consider a small disturbance, $\zeta(x, y, t)$, of a free surface of liquid from its equilibrium position $z = 0$ in a cylindrical tank of cross section A and depth h, as shown in Figure 1. The effects of viscosity being negligible in first approximation, the fluid motion will be irrotational, in consequence of which we may express the velocity at any point in the liquid as the negative gradient of a velocity potential ϕ. The compressibility of the liquid also being negligible, this potential must satisfy Laplace's equation

$$\nabla^2 \phi = 0 \tag{2.1}$$

The kinematic boundary conditions on ϕ, as dictated by the assumed velocity ζ_t at the free surface and the condition of zero normal velocity at the walls of the tank, are

$$-\phi_z = \zeta_t \quad \text{on} \quad z = 0 \tag{2.2a}$$

$$\phi_z = 0 \quad \text{on} \quad z = -h \tag{2.2b}$$

$$\phi_n = 0 \quad \text{on} \quad S \tag{2.2c}$$

where (2.2a) is imposed at the mean position of the free surface by virtue of the assumption of small displacement; S denotes the lateral area of the cylinder, and n is the inwardly directed normal to S.

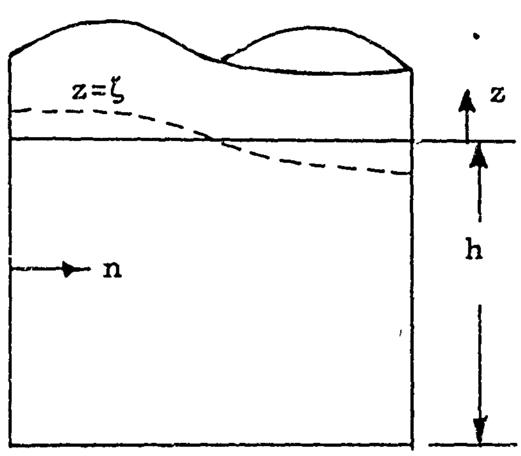

Figure 1. Cross Section of Cylindrical Tank

The solution to (2.1) and (2.2), due originally to Poisson[1] and Rayleigh,[2] may be exhibited in the form[3]

$$\zeta(x, y, t) = q(t)\ \psi(x, y) \tag{2.3}$$

$$\phi(x, y, z, t) = -\dot{q}(t)\ \psi(x, y) \left\{ \cosh\left[k(z+h)\right] / k \sinh(kh) \right\} \tag{2.4}$$

where ψ is a solution to the Helmholtz equation

$$\nabla^2 \psi + k^2 \psi = 0 \tag{2.5}$$

subject to the boundary condition

$$\psi_n = 0 \quad \text{on} \quad C \quad . \tag{2.6}$$

where C denotes the boundary of the cylinder in a plane of constant z.

The homogenous boundary-value problem presented by (2.5) and (2.6) is analogous to that presented by a vibrating membrane or propagation of sound in a cylindrical tube[4] and possesses a doubly infinite number of solutions, say ψ_{ij}, corresponding to those values of the parameter k, say k_{ij}, that are permitted by the boundary condition (2.6). The ψ_{ij}, as is well known, are orthogonal, so that the corresponding modes of motion are independent; accordingly, we may omit the subscripts with the implication that ψ denotes one mode only. We find it convenient, in the subsequent analysis, to render ψ dimensionless and normalize the individual modes such that

$$\iint \psi^2\, dA = A \tag{2.7}$$

We also have, from the consideration (dictated by continuity) that the mean elevation represented by ζ must vanish, the relation

$$\iint \psi\, dA = 0 \quad . \tag{2.8}$$

We emphasize in concluding this section that the formulation to this point rests on purely kinematical assumptions, viz., irrotationality and incompressibility.

SECTION III - KINETIC AND POTENTIAL ENERGIES

The function $q(t)$, dimensionally a length, may be regarded as a generalized coordinate for the particular mode (ψ and k) to which it corresponds. The kinetic energy of this mode, allowing for variations of depth with time, is given by

$$T = \frac{1}{2}\rho \iiint \left[\vec{k}\,\dot{h} - \nabla\phi\right]^2 dA\,dz \tag{3.1a}$$

$$= \frac{m}{2A}\iint dA \int_{-h}^{\zeta} \left[\dot{h}^2 - 2\dot{h}\,\phi_z + (\nabla\phi)^2\right] dz \tag{3.1b}$$

where ρ denotes the density of the liquid and m the mass per unit depth (i.e., $m = \rho A$). The individual contributions of the three terms in the integrand of (3.1b) may be evaluated according to

$$\frac{m}{2A}\iint dA \int_{-h}^{\zeta} \dot{h}^2\, dz = \frac{m\dot{h}^2}{2A}\iint (h+\zeta)\, dA = \frac{1}{2}\, m h \dot{h}^2 \tag{3.2}$$

$$\frac{m}{2A}\iint dA \int_{-h}^{\zeta} (-2\dot{h}\phi_z)\, dz = \frac{m\dot{h}}{A}\iint \left[\phi\Big|_{z=-h} - \phi\Big|_{z=0} - \phi_z\Big|_{z=0}\zeta + O(\zeta^3)\right] dA$$

$$= \frac{m\dot{h}}{A}\iint \zeta\zeta_t\, dA = m\dot{h}\,q\,\dot{q} \tag{3.3}$$

$$\frac{m}{2A}\iiint (\nabla\phi)^2\, dA\,dz = \frac{m}{2A}\iint_{z=0} \left[\phi\phi_z + O(\zeta^3)\right] dA = \frac{1}{2} m \left[\frac{\coth(kh)}{k}\right] \dot{q}^2 \tag{3.4}$$

where those integrals of $O(\zeta)$ vanish identically in virtue of (2.8) and those of $O(\zeta^2)$ have been evaluated with the aid of (2.7); the volume integral in (3.4) has been converted to a surface integral with the aid of Green's theorem, (2.1), and (2.2). The end result for the kinetic energy then reads

$$T - T_o = m\dot{h}\,q\,\dot{q} + \frac{1}{2} m \left[\frac{\coth(kh)}{k}\right] \dot{q}^2 \tag{3.5a}$$

$$T_o = \frac{1}{2}(mh)\,\dot{h}^2 \tag{3.5b}$$

where T_o is the kinetic energy associated solely with the changing level of the liquid, while $T - T_o$ represents the additional contribution of one mode only; more generally, the terms on the right-hand side of (3.5a) may be summed over all modes but do not involve cross products of the generalized coordinates.

The potential energy relative to the bottom of the tank is given by

$$V = \iint dA \int_{-h}^{\zeta} \rho g\ (z + h)\ dA \tag{3.6a}$$

$$= \frac{mg}{2A} \iint (h^2 + 2h\zeta + \zeta^2)\ dA \tag{3.6b}$$

Evaluating the integrals with the aid of (2.7) and (2.8), we obtain

$$V - V_o = \frac{1}{2}\ mg\ q^2 \tag{3.7a}$$

$$V_o = \frac{1}{2}\ mg\ h^2 \tag{3.7b}$$

where V_o denotes the potential energy of the undisturbed fluid. We remark that g, the acceleration of gravity, should be replaced by an equivalent value (say ng) if the entire tank is subjected to an acceleration.

We shall proceed further on the assumption that h varies slowly. The motion q(t) then will be approximately simple harmonic, and the mean values of the perturbation energies may be approximated by (neglecting the variation of h in averaging over one cycle)

$$\overline{T - T_o} = \frac{1}{2} m \left[\frac{\coth (kh)}{k}\right] \overline{\dot{q}^2} \tag{3.8}$$

$$\overline{V - V_o} = \frac{1}{2}\ mg\ \overline{q^2} \tag{3.9}$$

SECTION IV - VISCOUS DISSIPATION

We shall calculate the frictional dissipation of the free oscillations on the assumption that viscous forces are negligible everywhere except in a laminar boundary layer at the walls of the tank, reflecting the well-known fact[5] that the logrithmic decrements associated with friction inside and outside of this layer are $0(\nu/\omega R^2)^{1/2}$ and $0(\nu/\omega R^2)^1$, respectively, where ν is the kinematic viscosity, ω the angular frequency, and R a characteristic length proportional to area/perimeter (often called the hydraulic radius). The thickness of the boundary layer being small compared with all other characteristic lengths, we may also treat the viscous flow as plane and neglect the pressure gradient. The linearized equation for the velocity $\vec{v}$ in the boundary layer then contains only the inertial and shear terms and reads[6]

$$\frac{\partial \vec{v}}{\partial t} = \nu \frac{\partial^2 \vec{v}}{\partial n^2} \tag{4.1}$$

We require a solution to this equation that vanishes outside the boundary layer ($n \longrightarrow \infty$) and just cancels the tangential velocity, say $\vec{v}_o \cos(\omega t + \epsilon)$, calculated at the wall in the absence of friction, viz.,

$$\vec{v} \longrightarrow 0 \quad , \quad n \longrightarrow \infty \tag{4.2a}$$

$$\vec{v} = \vec{v}_o \cos(\omega t + \epsilon) \quad , \quad n = 0 \tag{4.2b}$$

The problem posed by (4.1) and (4.2) corresponds to that for an oscillating flat plate, first solved by Stokes[7] with the result[8]

$$\vec{v}' = e^{-\beta n} \vec{v}_o \cos(\omega t - \beta n + \epsilon) \tag{4.3a}$$

$$\beta = (\omega/2\nu)^{1/2} \tag{4.3b}$$

The integrated flow defect per unit width of the boundary layer is

$$\int_{o}^{\infty} \vec{v} \; dn = \frac{\vec{v}_o \cos(\omega t - \pi/4)}{\sqrt{2}\,\beta} \tag{4.4}$$

We infer from this result that the mean flow defect in the boundary layer lags the outer flow by 45^o, and that the displacement thickness is

$$\delta_1 = \frac{1}{\sqrt{2}\,\beta} \approx \sqrt{\frac{\nu}{\omega}} \tag{4.5}$$

The corresponding Reynolds number is

$$R_1 = \frac{\left|\vec{v}_o \; \delta_1\right|}{\sqrt{2}\,\nu} = \frac{\left|\vec{v}_o\right|}{\sqrt{2\omega\nu}} = \frac{A}{\delta_1} \tag{4.6}$$

where $\left|\vec{v}_o/\sqrt{2}\right|$ and $A(=\omega^{-1}\left|\vec{v}_o/\sqrt{2}\right|)$ denote the r.m.s. velocity and amplitude of the motion outside the boundary layer.

The question of the stability of the laminar flow given by (4.3) does not appear to have been examined in the literature, but we may approximate the amplitude of motion necessary to initiate turbulence by comparison to the transition problem for steady flow along a semi-infinite flat plate. The minimum transition Reynolds number for the steady flow problem, based on displacement thickness,* is found experimentally to vary between about 500 and 1700, depending on the level of free stream turbulence[10] (the theoretical minimum value is about 420[11]). Taking the lowest experimental value, we infer that the oscillatory flow in the boundary layer will remain laminar if

$$A < 500\,\delta_1 = 500\sqrt{\frac{\nu}{\omega}} \tag{4.7}$$

*We remark that this transition Reynolds number is rather insensitive to the pressure gradient in the outer flow[9], from which we surmise that the details of the outer flow are sufficiently unimportant to warrant a moderate expectation of success in the extrapolation to an oscillatory flow.

This result also may be obtained, at least as to order of magnitude, on the basis of a transition time (equal to transition distance divided by free stream velocity for the steady flow problem) Reynolds number if the allowable time in the oscillatory problem is equated to some fraction of the period; the numerical value of (4.7) is obtained if this fraction is assumed to correspond to a phase shift of approximately 45°.

The viscous shear in the boundary layer is given by $-\mu\,(\partial \vec{v}/\partial n)$, so that the rate of dissipation per unit area is, in terms of a dissipation function F[12],

$$2\frac{\partial F}{\partial S} = \mu \int_0^{\infty} \left| \frac{\partial \vec{v}}{\partial n} \right|^2 dn \tag{4.8}$$

We require only the mean value of this dissipation function, viz.,

$$2\frac{\partial \overline{F}}{\partial S} = \mu\beta^2 \left| \vec{v}_o \right|^2 \int_0^{\infty} e^{-2\beta n} \overline{\left[\cos(\omega t - \beta n + \epsilon) - \sin(\omega t - \beta n + \epsilon)\right]^2}\, dn \tag{4.9a}$$

$$= \mu\beta^2 \left| \vec{v}_o \right|^2 \int_0^{\infty} e^{-2\beta n}\, dn \tag{4.9b}$$

$$= \frac{1}{2}\,\mu\beta \left| \vec{v}_o \right|^2 \tag{4.9c}$$

We consider first the integral of $\left| \vec{v}_o \right|^2$ over the bottom, where

$$\left| \vec{v}_o \right|^2 \Big|_{z=-h} = 2\,\overline{\dot{q}^2}\,(\nabla\psi)^2/k^2\,\mathrm{sh}^2(kh) \tag{4.10}$$

the factor of two accounting for the fact that the mean square of $\vec{v}$ is one-half the square of its amplitude in consequence of the assumed harmonic motion. Neglecting the variation of depth over one cycle, we then have

$$\iint_{z=-h} \left| \vec{v}_o \right|^2 dA = \frac{2\,\overline{\dot{q}^2}}{k^2\,\mathrm{sh}^2(kh)} \iint (\nabla\psi)^2\, dA \tag{4.11a}$$

$$= \frac{-2\,\overline{\dot{q}^2}}{k^2\,\mathrm{sh}^2(kh)} \iint \psi\nabla^2\psi\, dA \tag{4.11b}$$

$$= 2A\,\overline{\dot{q}^2}\,\mathrm{csch}^2(kh) \tag{4.11c}$$

where the integral has been reduced with the aid of Green's theorem and (2.5) through (2.7). Similarly, the integral over the lateral wall is

$$\iint \left|\vec{v}_o\right|^2 dS = \frac{2\overline{q^2}}{sh^2(kh)} \oint d\ell \int_{-h}^{o} \left\{ \frac{ch^2\left[k(z+h)\right]}{k^2} \left(\frac{\partial\psi}{\partial\ell}\right)^2 + sh^2\left[k(z+h)\right]\psi^2 \right\} dz$$

$$= \overline{q^2} \oint \left\{ \left[\frac{\coth(kh)}{k} + h\,\mathrm{csch}^2(kh)\right] \left(\frac{1}{k}\frac{\partial\psi}{\partial\ell}\right)^2 \right. \tag{4.12a}$$

$$\left. + \left[\frac{\coth(kh)}{k} - h\,\mathrm{csch}^2(kh)\right]\psi^2 \right\} d\ell \tag{4.12b}$$

Adding (4.12b) to (4.11c) and multiplying the total by $\mu\beta/2$, the total dissipation per cycle may be placed in the form

$$2F = 2\,a\,m\,k^{-1}\coth(kh)\,\overline{q^2} \tag{4.13}$$

where

$$a = \frac{1}{2}\sqrt{\frac{\nu\omega}{2}}\left[\frac{1}{R_1} + \frac{2\,kh\,\mathrm{csch}\,(2kh)}{R_2}\right] \tag{4.14}$$

$$\frac{1}{R_1} = \frac{1}{A}\oint\left[\psi^2 + \left(\frac{1}{k}\frac{\partial\psi}{\partial\ell}\right)^2\right]d\ell \tag{4.15a}$$

$$\frac{1}{R_2} = \frac{1}{h} - \frac{1}{A}\oint\left[\psi^2 - \left(\frac{1}{k}\frac{\partial\psi}{\partial\ell}\right)^2\right]d\ell \tag{4.15b}$$

We may designate the lengths R_1 and R_2, which are proportional to wetted area divided by perimeter, as modified hydraulic radii.

SECTION V - FREQUENCY AND DECREMENT OF MEAN MOTION

The mean (over one cycle) motion of the approximately harmonic motion will be governed by the equation

$$\frac{\partial}{\partial t}\left[(\overline{T - T_o}) + (\overline{V - V_o})\right] = -2\overline{F} \tag{5.1}$$

provided that the change over one cycle is small. The mean values of the perturbation energies are given by (3.8) and (3.9), while the dissipation function is given by (4.9); noting that

$$\overline{\dot{q}^2} = \omega^2 \; \overline{q^2} \tag{5.2}$$

for an approximately harmonic motion, these means become

$$\overline{T - T_o} = \frac{1}{2} \; m\omega^2 \; k^{-1} \; \coth (kh) \; \overline{q^2} \tag{5.3}$$

$$\overline{V - V_o} = \frac{1}{2} \; mg \; \overline{q^2} \tag{5.4}$$

$$2\overline{F} = 2 \; \alpha \; m\omega^2 k^{-1} \coth (kh) \, \overline{q^2} \tag{5.5}$$

Substituting (5.3) through (5.5) in (5.1) and dividing through by mg, we obtain

$$\frac{1}{2}\left[\left(\frac{\omega^2}{kg}\right)\coth (kh) + 1\right]\frac{\partial \overline{q^2}}{\partial t} + \left[\left(\frac{2\alpha\,\omega^2}{kg}\right)\coth (kh) - \left(\frac{\omega^2 \dot{h}}{2g}\right) \operatorname{csch}^2 (kh)\right] \overline{q^2} = 0 \tag{5.6}$$

We may obtain an adequate approximation to the frequency in (5.6) by noting that the mean values of kinetic and potential energy would be exactly equal for a truly harmonic motion, whence

$$\omega^2 = kg \tanh (kh) \tag{5.7}$$

in agreement with the classical result for constant depth.[13] The time dependence of ω, as given by (5.7), and the deviations from this value associated with viscous dissipation are both second-order effects in (5.6), so that we may substitute (5.7) directly in (5.6) to obtain

$$\frac{\partial \overline{q^2}}{\partial t} + \left[2\alpha - k\dot{h} \;\operatorname{csch}(2kh)\right] \overline{q^2} = 0 \tag{5.8}$$

which yields

$$\overline{q^2} = \overline{q_o^2} \;\exp\left\{ - \left[2\alpha - k\dot{h} \;\operatorname{csch}(2kh)\right] t \right\} \tag{5.9}$$

where $\overline{q_o^2}$ is the initial, mean square value of q.

The damping ratio (for the r.m.s. amplitude) implied by (5.9) is

$$\gamma = \omega^{-1} \left[\alpha - \frac{1}{2} \; k\dot{h} \operatorname{csch}(2kh) \right] \tag{5.10}$$

Substiuting α from (4.8), we have

$$\gamma = \frac{1}{2} \sqrt{\frac{\nu}{2\omega R_1 2}} + kh \operatorname{csch}(2kh) \left[\sqrt{\frac{\nu}{2\omega R_2^2}} - \frac{\dot{h}}{2\omega h} \right] \tag{5.11}$$

We note that time varying depth has a destabilizing or stabilizing effect as $\dot{h}$ is positive or negative, respectively. The magnitude of the effect is, however, rather small for practical values of $(\dot{h}/\omega h)$.

In most applications kh will be sufficiently large (say kh > 2) to justify the neglect of terms of order e^{-2kh}, and the results (5.7) and (5.11) simplify to

$$\omega^2 = kg \left[1 + O(e^{-2kh}) \right] \tag{5.12}$$

$$\gamma = \frac{1}{2} \sqrt{\frac{\nu}{2\sqrt{kg}\, R_1 2}} \left[1 + O(e^{-2kh}) \right] \tag{5.13}$$

In this approximation the effect of varying depth is negligible.

SECTION VI - PRESSURE DISTRIBUTION

The pressure distribution (p) on the lateral walls of the tank may be calculated from the Bernoulli equation

$$p - p_o = \rho(\phi_t - gz) \tag{6.1}$$

where p_o denotes the pressure above the free surface (the effects of friction and changing depth on the pressure are neglected). The time derivative of the potential, as given by (2.4), is

$$\phi_t = -\zeta_{tt}\left\{ch\left[k(z+h)\right]/k\ sh(kh)\right\} \tag{6.2a}$$

$$= \omega^2 \zeta \left\{ch\left[k(z+h)\right]/k\ sh(kh)\right\} \tag{6.2b}$$

$$= g\zeta\left\{ch\left[k(z+h)\right]/ch(kh)\right\} \tag{6.2c}$$

where, in (6.2c), ω^2 has been substituted from (5.7). Substituting (6.2c) in (6.1), we obtain

$$p - p_o = \rho g\left\{\frac{\zeta ch\left[k(z+h)\right]}{ch(kh)} - z\right\} \tag{6.3}$$

We remark that, as required, $p = p_o$ at $z = \zeta$ (neglecting terms of order ζ^2); indeed, this requirement affords the usual derivation of (5.7).

The integrated pressure on a vertical strip of the tank is given by

$$\int_{-h}^{\zeta}(p - p_o)\,dz = \rho g\left\{\frac{\zeta\ sh\left[k(z+h)\right]}{k\ ch(kh)}\Bigg|_{-h}^{\zeta} + \frac{1}{2}(h^2 - \zeta^2)\right\} \tag{6.4a}$$

$$= \frac{1}{2}\rho g h^2 + \rho g \zeta\ k^{-1}\tanh(kh) + O(\zeta^2) \tag{6.4b}$$

The corresponding moment with respect to the bottom of the tank is given by

$$\int_{-h}^{\zeta} (p - p_o)(z+h)\, dz = \frac{1}{6}\rho g h^3 + \rho g \zeta \left\{ k^{-1} h \tanh (kh) + k^{-2} \left[\operatorname{sech} (kh) - 1 \right] \right\} + O(\zeta^2) \tag{6.5a}$$

$$= \frac{1}{6}\rho g h^3 + \rho g \zeta k^{-1} \tanh (kh) \cdot \left[h - k^{-1} \tanh (kh/2) \right] \tag{6.5b}$$

The center of pressure of the perturbation (ζ) component is, therefore, a distance

$$d = h - k^{-1} \tanh (kh/2) \tag{6.6}$$

from the bottom of the tank.

SECTION VII - CIRCULAR TANK

We consider now the application of the foregoing results to the more specific configuration of a circular cylinder of radius a. The approximate solutions of (2.5) and (2.6) read, in the cylindrical polar coordinates r and θ,[14]

$$\psi(r, \theta) = C_{sk} J_s (kr) \cos (s\theta) , \qquad s = 0, 1, 2, \ldots \tag{7.1}$$

where J_s denotes Bessel's function of order s, and k is any root to the transcendental equation

$$J_s' (ka) = 0 \tag{7.2}$$

The normalizing coefficient C_{sk} is determined by the requirement (2.7), viz.,

$$1 = \frac{1}{A} \iint \psi^2 \, dA = \frac{C_{sk}^2}{\pi a^2} \int_o^a \int_o^{2\pi} J_s^2 (kr) \cos^2 (s\theta) \, r \, dr \, d\theta$$

$$\doteq C_{sk}^2 (1 + \delta_s^o) \left(\frac{k^2 a^2 - s^2}{2 k^2 a^2} \right) J_s^2 \; (ka) \tag{7.3}$$

The dominant mode of oscillation corresponds to the smallest root of (7.2) and is specified by

$$s = 1 , \qquad ka = 0.586 \pi = 1.84 \tag{7.4}$$

The corresponding frequency, as determined by (5.7), is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{kg \tanh (kh)} \tag{7.5a}$$

$$= 0.216 \sqrt{\frac{g}{a} \tanh (1.84 \, h/a)} \tag{7.5b}$$

The (reciprocal) hydraulic radii, obtained by substituting (7.1) and (7.3) in (4.15a, b), are found to be

$$\frac{1}{R_1} = \frac{2}{a} \left(\frac{k^2 a^2 + s^2}{k^2 a^2 - s^2}\right) \tag{7.6a}$$

$$\frac{1}{R_2} = \frac{1}{h} - \frac{2}{a} \tag{7.6b}$$

The corresponding damping ratio, obtained by substituting (7.5a) and (7.6a, b) in (5.11) and neglecting $(\dot{h}/\omega h)$, is

$$\gamma = \left(\frac{\nu}{2}\right)^{1/2} a^{-3/4} g^{-1/4} (ka)^{-1/4} \tanh^{1/4} (kh) \cdot \left[\left(\frac{k^2 a^2 + s^2}{k^2 a^2 - s^2}\right) + (ka - 2kh) \operatorname{csch} (2kh)\right] \tag{7.7}$$

which reduces further to

$$\gamma = 1.12 \ \nu^{1/2} a^{-3/4} g^{-1/4} \left[1 + (1 - \frac{2h}{a}) \operatorname{csch} (3.68h/a)\right] \cdot \tanh^{1/4} (1.84h/a) \tag{7.8}$$

for the dominant mode of (7.4).

The depth correction functions for frequency and damping ratio, as given by (7.5) and (7.8), are plotted as functions of the ratio of depth to radius in Figure 2 and are seen to approximate unity rather closely when the depth exceeds the radius and to differ therefrom by a negligible amount when the depth exceeds the diameter.

We consider as a numerical example a tank 10 feet in diameter and of equal or greater depth. The period given by (7.5b) is 1.81 seconds, while the damping ratio given by (7.7) for a liquid (e.g., water at 20°C) having a kinematic viscosity of 10^{-2} stokes is 4.6×10^{-4}; the latter figure is equivalent to a time of 7 minutes to damp to one-half amplitude.

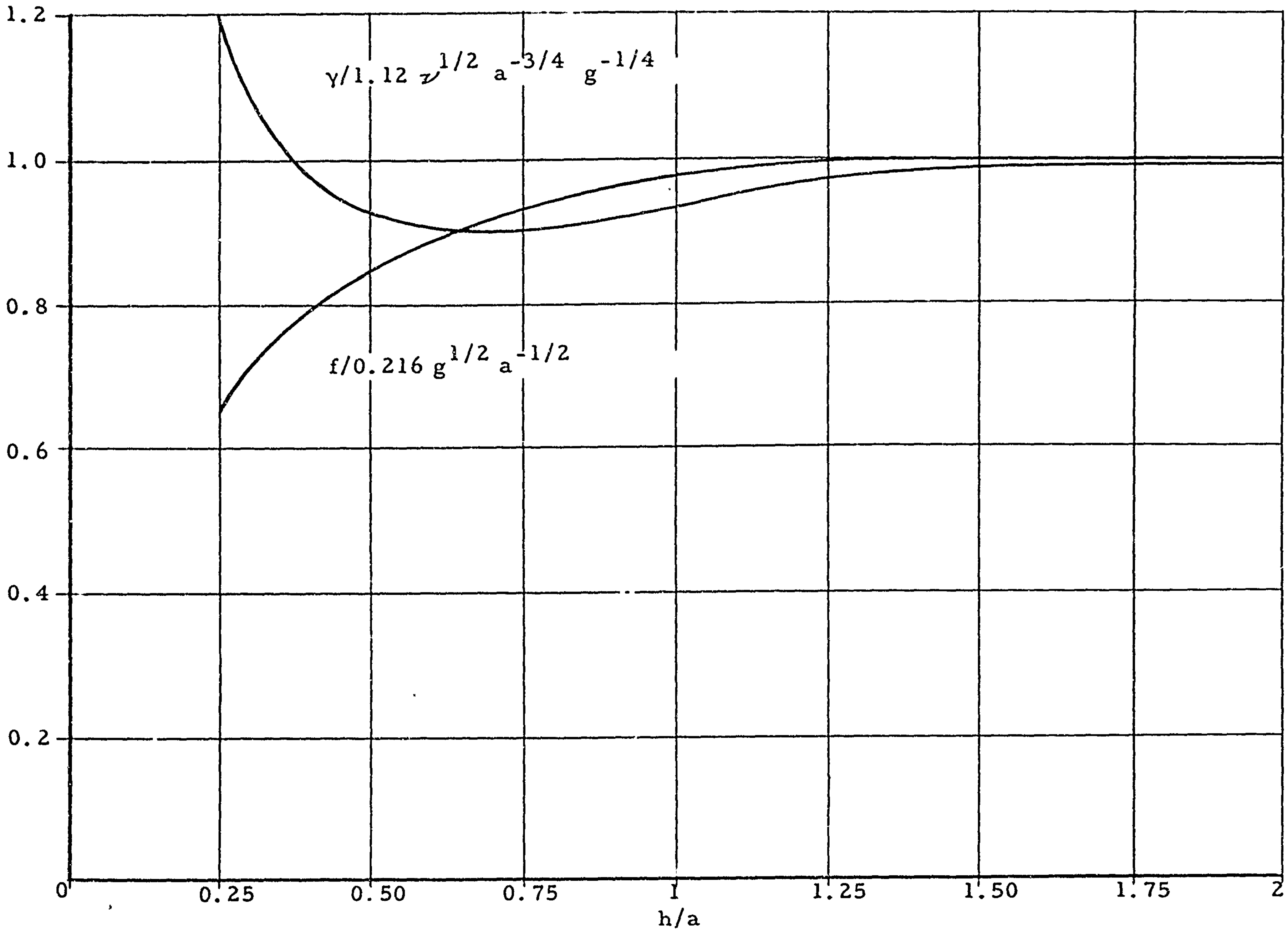

Figure 2. Finite Depth Corrections for Frequency and Damping Ratio of Dominant Mode in Circular Tank, as given by (7.5b) and (7.8)

The displacement thickness given by (4.5) is 0.054 cm, and the corresponding minimum amplitude necessary to render the laminar flow unstable is, according to (4.7), 27 cm (10.6 inches). We note that this must be interpreted as a (spacewise) mean amplitude and that it is based on the minimum transition Reynolds number.

The perturbation pressure on a vertical elemen f the wall of a circular tank may be calculated from (6.4b). The corresponding transverse force on the tank is given by

$$F = \rho g k^{-1} \tanh (kh) \int_{o}^{2\pi} \zeta \cos \theta \cdot a d\theta \tag{7.9a}$$

$$= \rho g \pi a^2 \, |\zeta| (ka)^{-1} \tanh (kh) \cos (\omega t) \tag{7.9b}$$

where $|\zeta|$ denotes the maximum amplitude at the wall, and the phase angle has been arbitrarily prescribed as zero. Assuming the dominant mode of (7.4), (7.9b) yields a maximum force of 2700 lb (oscillatory) for a sloshing of one-foot amplitude in a 10-foot tank of water. This force, according to (6.6), would act approximately 2.7 feet below the mean surface.

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